Section 8.3

8. Let $f : \mathbb{R} \to \mathbb{R}$ be such that $f'(x) = f(x)$ for all $x \in \mathbb{R}$. Show that there exists $K \in \mathbb{R}$ such that $f(x) = Ke^x$ for all $x \in \mathbb{R}$.

This is a direct consequence of Theorem 8.3.4

By the argument in the proof of Theorem 8.3.4,

if $f(0)=0$, then $f \equiv 0$

Hence, we can assume $f(0) \neq 0$ and consider $g(x) = \dfrac{f(x)}{f(0)}$.

which satisfies $g(0) = 1$ and $g'(x) = g(x)$, $\forall x \in \mathbb{R}$

Again by Theorem 8.3.4 and Definition 8.3.5,

$g(x) = e^x$, $\forall x \in \mathbb{R}$

Then $f(x) = Ke^x$ where we let $K = f(0)$

Section 8.4

8. If $f : \mathbb{R} \to \mathbb{R}$ is such that $f''(x) = f(x)$ for all $x \in \mathbb{R}$, show that there exist real numbers α, β such that $f(x) = \alpha c(x) + \beta s(x)$ for all $x \in \mathbb{R}$. Apply this to the functions $f_1(x) := e^x$ and $f_2(x) := e^{-x}$ for $x \in \mathbb{R}$. Show that $c(x) = \frac{1}{2}(e^x + e^{-x})$ and $s(x) = \frac{1}{2}(e^x - e^{-x})$ for $x \in \mathbb{R}$.

As is implied in Q6, define two sequences of function $\{C_n\}$ and $\{S_n\}$ inductively by

$$\begin{cases} C_1(x) = 1 \quad , \quad S_1(x) = x \\ S_n(x) = \int_0^x C_n(x)\, dx \\ C_{n+1}(x) = 1 + \int_0^x S_n(x)\, dx \end{cases}$$

Then by the same argument as in the proof of Theorem 8.4.1, there exist functions $C(x)$ and $S(x)$, s.t. $C(0)=1$, $C'(0)=0$, $S(0)=0$, $S'(0)=1$ and $C''(x) = C(x)$, $S''(x) = S(x)$, $C'(x) = S(x)$, $S'(x) = C(x)$

where $C(x) = \lim_{n\to\infty} C_n(x)$, $S(x) = \lim_{n\to\infty} S_n(x)$

And by the uniqueness argument in Theorem 8.4.4, we can also derive $C(x)$ and $S(x)$ are unique in the sense that if $h(0) = h'(0) = 0$ and $h''(x) = h(x)$, $\forall x \in \mathbb{R}$, then $h \equiv 0$

Now let $g(x) := f(0)\, C(x) + f'(0)\, S(x)$

Then $g'(x) = f(0)\, S(x) + f'(0)\, C(x)$

We have $g(0) = f(0)$, $g'(0) = f'(0)$

Take $\varphi(x) := f(x) - g(x)$, then $\varphi(0) = \varphi'(0) = 0$ and $\varphi''(x) = \varphi(x)$

By uniqueness, $\varphi \equiv 0$, which implies $f(x) = \alpha C(x) + \beta S(x)$

where $\alpha = f(0)$, $\beta = f'(0)$

Take $f_1(x) = e^x$ and $f_2(x) = e^{-x}$, we get

$$\begin{cases} e^x = c(x) + s(x) \\ e^{-x} = c(x) - s(x) \end{cases}$$

Hence, $c(x) = \frac{1}{2}(e^x + e^{-x})$, $s(x) = \frac{1}{2}(e^x - e^{-x})$

Section 9.1

2. Show that if a series is conditionally convergent, then the series obtained from its positive terms is divergent, and the series obtained from its negative terms is divergent.

A sequence $\{a_n\}$ is said to be conditionally convergent if $\sum_{n=1}^{\infty} a_n$ is convergent but $\sum_{n=1}^{\infty} |a_n|$ is divergent

Let $p_n = \frac{a_n + |a_n|}{2} = \begin{cases} a_n & \text{if } a_n > 0 \\ 0 & \text{otherwise} \end{cases}$

$q_n = \frac{a_n - |a_n|}{2} = \begin{cases} a_n & \text{if } a_n < 0 \\ 0 & \text{otherwise} \end{cases}$

Moreover, $|a_n| = 2p_n - a_n = a_n - 2q_n$

Then $\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} (2p_n - a_n) = \sum_{n=1}^{\infty} (a_n - 2q_n)$

Since $\sum_{n=1}^{\infty} |a_n|$ diverges but $\sum_{n=1}^{\infty} a_n$ is convergent, both $\sum_{n=1}^{\infty} p_n$ and $\sum_{n=1}^{\infty} q_n$ are divergent

Section 9.1

7. (a) If $\sum a_n$ is absolutely convergent and (b_n) is a bounded sequence, show that $\sum a_n b_n$ is absolutely convergent.

   (b) Give an example to show that if the convergence of $\sum a_n$ is conditional and (b_n) is a bounded sequence, then $\sum a_n b_n$ may diverge.

(a) Since $\{b_n\}$ is bounded, we can find a large $L > 0$ s.t.

$|b_n| < L$ for all n

Then $\sum_{n=1}^{\infty} |a_n b_n| < L \sum_{n=1}^{\infty} |a_n| < \infty$

Moreover, the sum $\sum_{i=1}^{n} |a_i b_i|$ is monotonely increasing

Hence $\sum_{n=1}^{\infty} |a_n b_n|$ converges

(b) Consider $a_n := \frac{(-1)^n}{n}$, $b_n := (-1)^n$

which satisfies the condition

But $\sum_{n=1}^{\infty} a_n b_n = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges